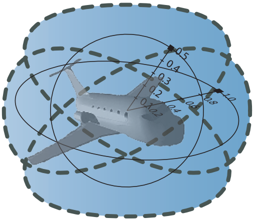
Differential Invariants for Collision Avoidance

André Platzer   Edmund M. Clarke

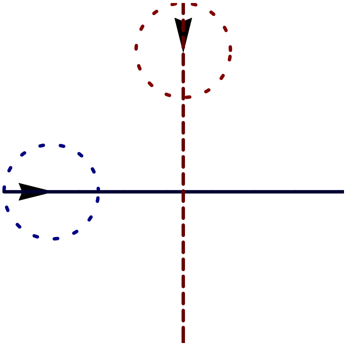
Carnegie Mellon University, Computer Science Department, Pittsburgh, PA

NSF CMACS Expedition



Outline

Air Traffic Control: Straight Lines & Instant Turns



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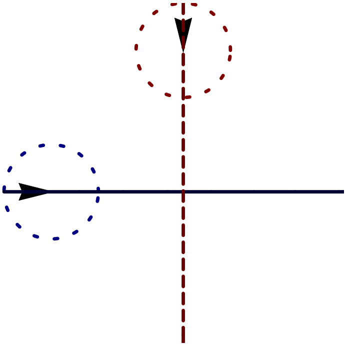

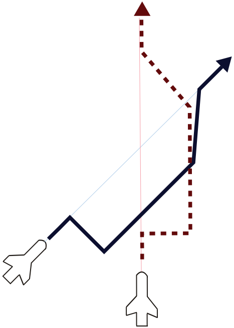

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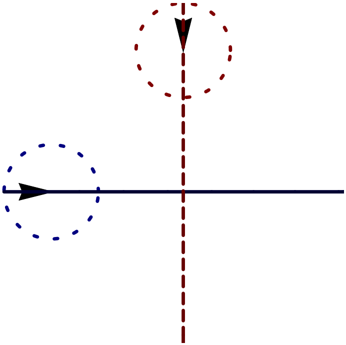
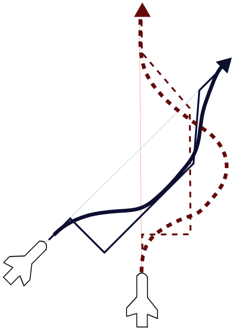
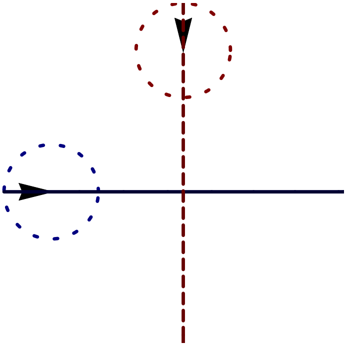




Air Traffic Control: Hybrid Systems & Curves



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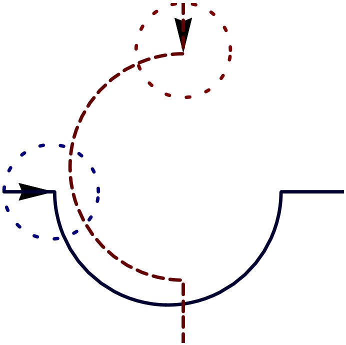

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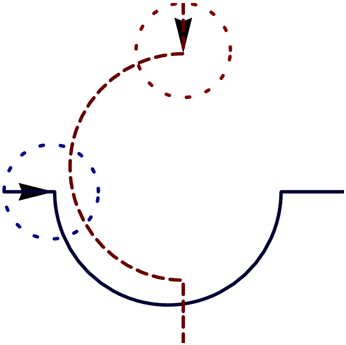
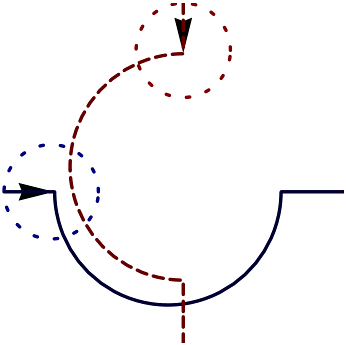
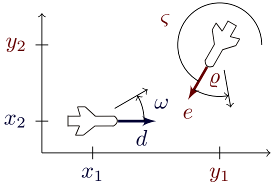


Hybrid Systems

continuous evolution along differential equations + discrete change





$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varrho - \omega \end{bmatrix}$$

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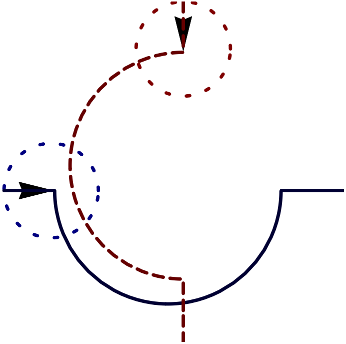

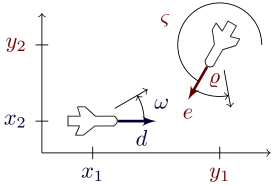

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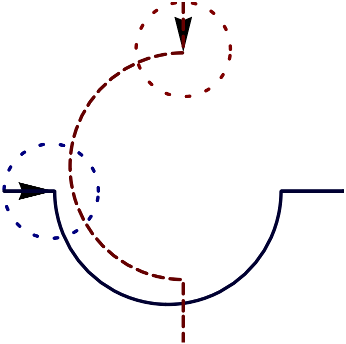
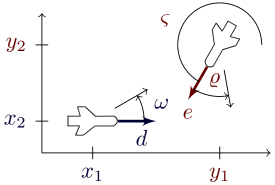
Example ("Solving" differential equations)

$$x_1(t) = \frac{1}{\omega\varrho}\big(x_1\omega\varrho\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varrho\sin\vartheta - v_1\varrho\sin t\omega$$
$$+ x_2\omega\varrho\sin t\omega - v_2\omega\cos\vartheta\cos t\varrho\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varrho + v_2\omega\sin\vartheta\sin t\omega\sin t\varrho\big)\ldots$$

Air Traffic Control: Hybrid Systems & Curves





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Example ("Solving" differential equations)

$$\forall t \geq 0 \quad \frac{1}{\omega\varrho}\Big(x_1\omega\varrho\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varrho\sin\vartheta - v_1\varrho\sin t\omega$$

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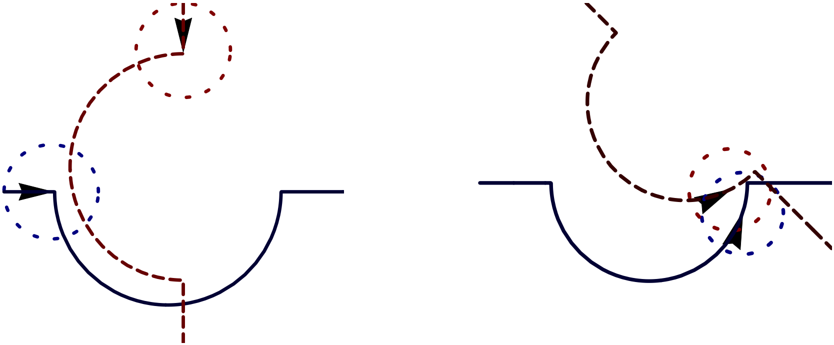

Hybrid Systems

continuous evolution along differential equations + discrete change

Introduce: Flyable Roundabout Maneuver

Problem ⇒ Solution

- Unrealistic instant turns can cause problems

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Problem $\Rightarrow$ Solution

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$\Rightarrow$ Introduce smoothly curved flyable maneuver as hybrid system model

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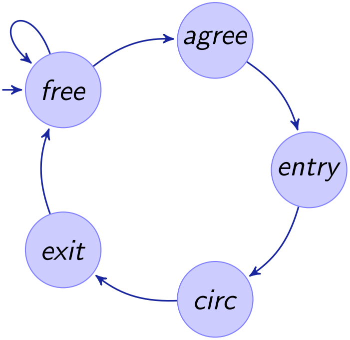

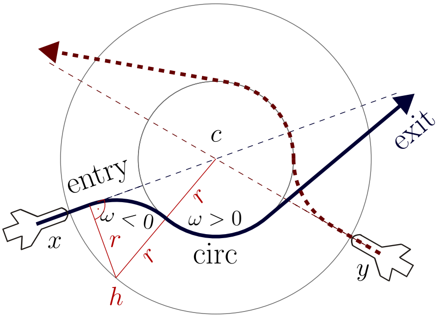

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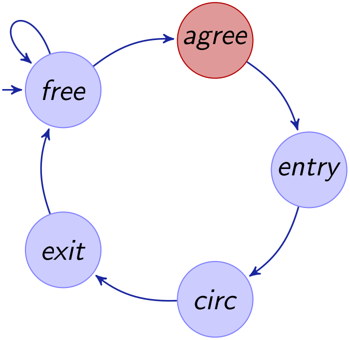

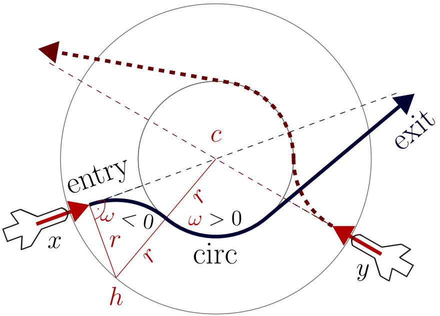

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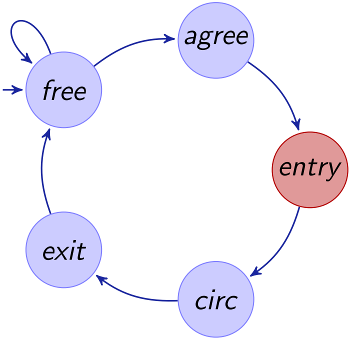

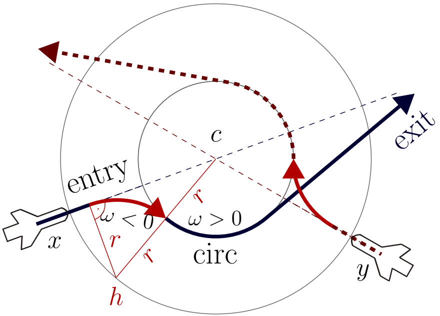

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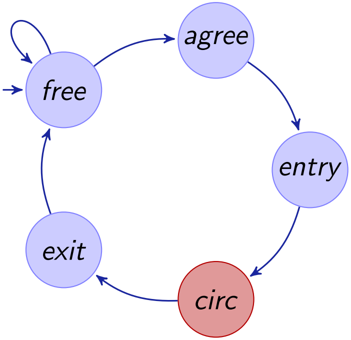

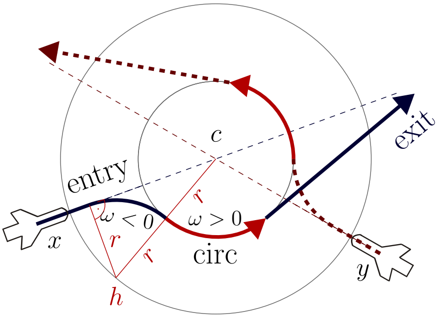

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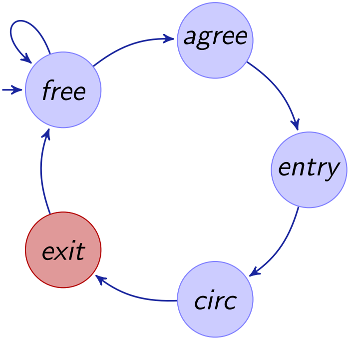

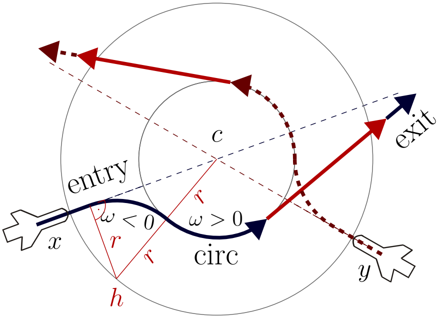

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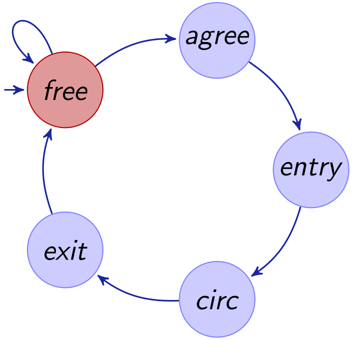

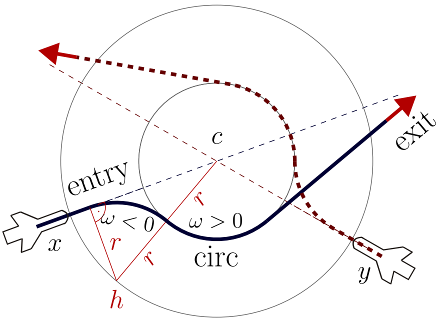

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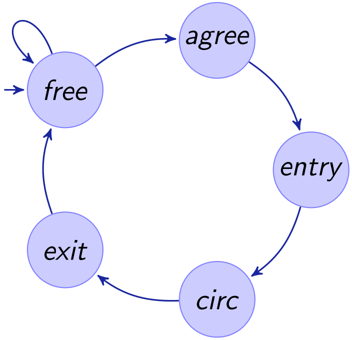

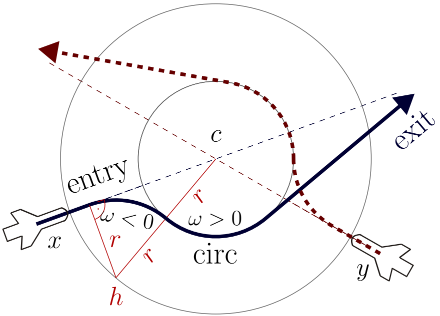

Problem $\Rightarrow$ Solution

- Unrealistic instant turns can cause problems ($\Rightarrow$ smooth curves)
- Geometric intuition can be misleading ($\Rightarrow$ hybrid system model)

$\Rightarrow$ Introduce smoothly curved flyable maneuver as hybrid system model

Verification for: nonlinear curve dynamics + mode switching?

Outline

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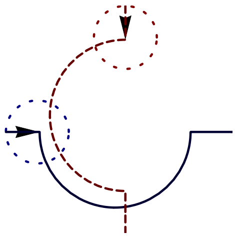
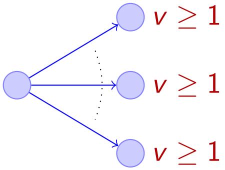
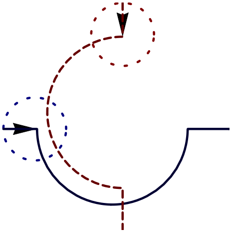


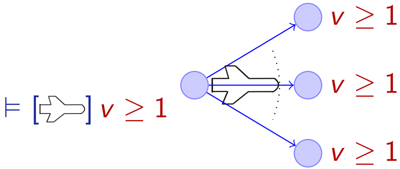
$\models \Box\, v \geq 1$



differential dynamic logic

$d\mathcal{L} = FOL_{\mathbb{R}} + DL$

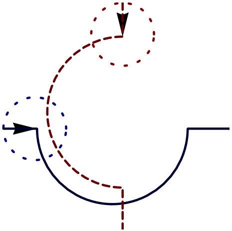




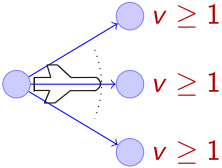
Logic for Hybrid Programs

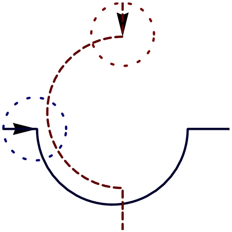
differential dynamic logic

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$



$\vDash [d_1' = -\omega d_2, d_2' = \omega d_1]\, v \geq 1$

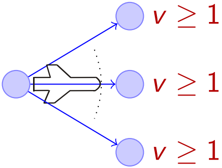




differential dynamic logic

$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{DL} + \text{HP}$

$\vDash [\texttt{if}(x_1 > 0)\, \omega := 1;\ d_1' = -\omega d_2, d_2' = \omega d_1]\, v \geq 1$



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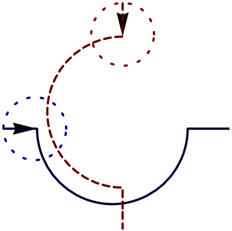

differential dynamic logic

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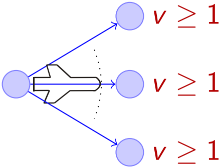
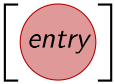
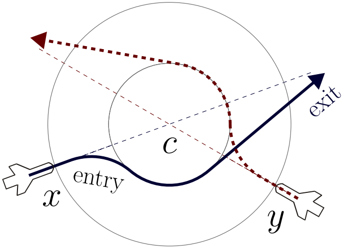
$$\vDash [\underbrace{\texttt{if}(x_1 > 0)\,\omega := 1;\; d_1' = -\omega d_2, d_2' = \omega d_1}_{\text{hybrid program}}]\, v \geq 1$$



Logic for Compositional Verification



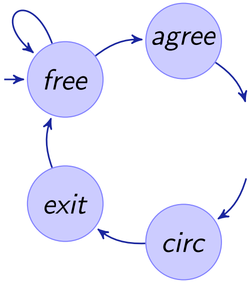


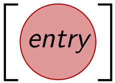
Example

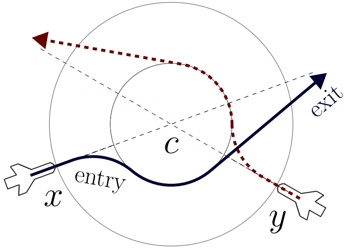
$$\mathit{safe} \land \mathit{far} \quad \rightarrow \quad [\mathit{entry}](\mathit{safe} \land \mathit{tangential})$$

$$\text{where} \quad \mathit{safe} \quad \equiv (x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2$$

Logic for Compositional Verification



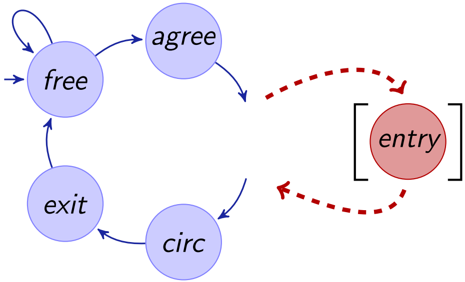


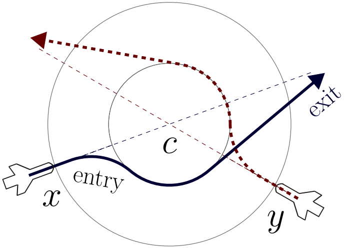


Example

$$\begin{aligned} \textit{safe} \wedge \textit{far} &\rightarrow [\textit{entry}](\textit{safe} \wedge \textit{tangential}) \\ \textit{safe} \wedge \textit{tangential} &\rightarrow [\textit{other subsystem}]\textit{safe} \\ \text{where} \quad \textit{safe} &\equiv (x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2 \end{aligned}$$

Logic for Compositional Verification





Example

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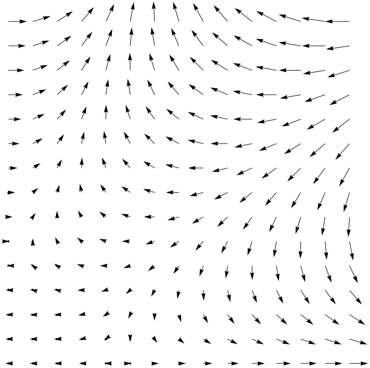
} conjunction

Differential Invariants for Differential Equations

"Definition" (Differential Invariant)

"Formula that remains true in the direction of the dynamics"



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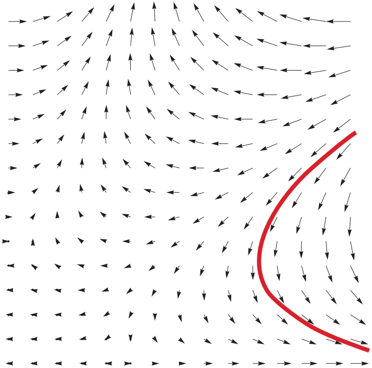

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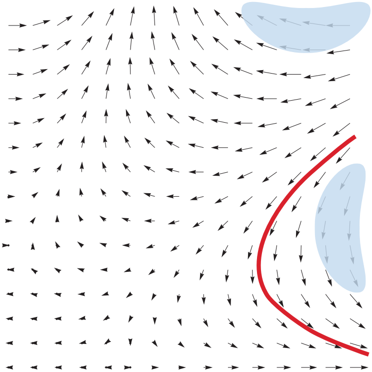
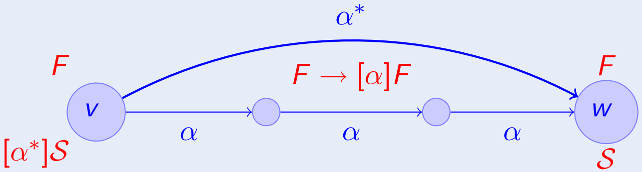


Discrete versus Differential Invariants

Definition (Discrete Invariant F)



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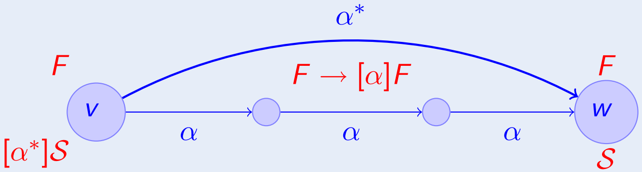

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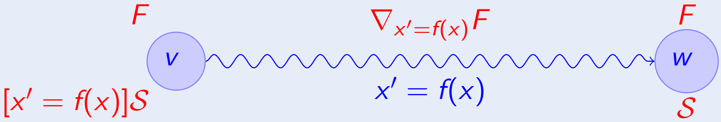
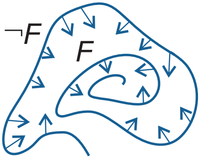
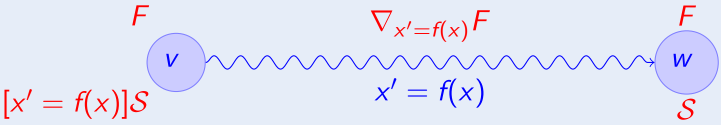


Discrete versus Differential Invariants

$$\nabla_{x_1'=f_1(x),\dots,x_n'=f_n(x)} F \quad \text{is} \quad \bigwedge_{(b\geq c)\in F} \left(\sum_{i=1}^{n} \frac{\partial b}{\partial x_i} f_i(x) \geq \sum_{i=1}^{n} \frac{\partial c}{\partial x_i} f_i(x) \right)$$

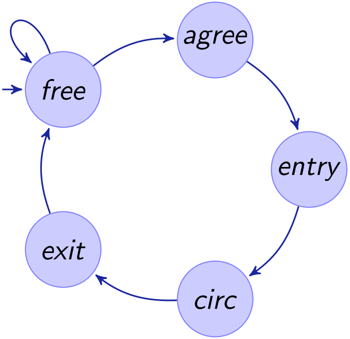


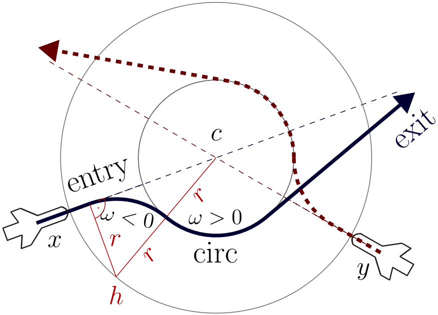
Definition (Differential Invariant F)



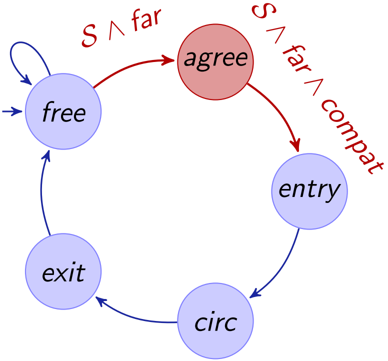
Outline

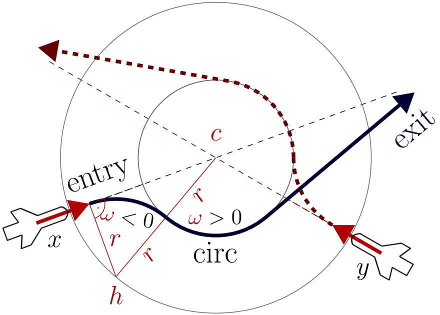
Verification Loop for Air Traffic Control





Verification Loop for Air Traffic Control

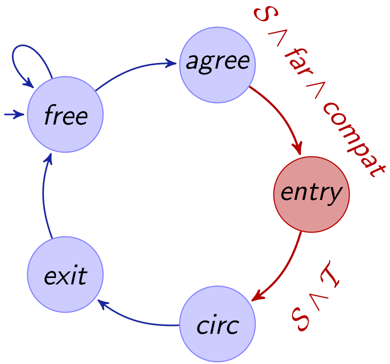


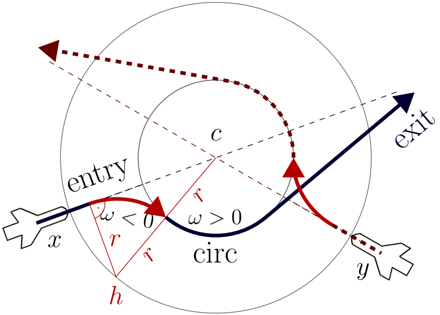


Example (dℒ formula of verification subgoal)

$$\textit{safe} \land \textit{far} \;\rightarrow\; [\textit{agree}](\textit{safe} \land \textit{far} \land \textit{compatible})$$

Verification Loop for Air Traffic Control

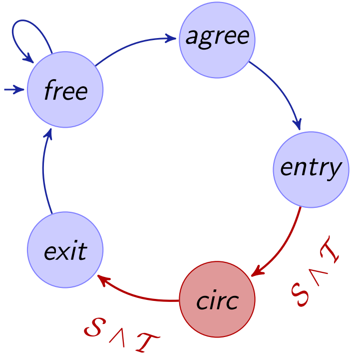


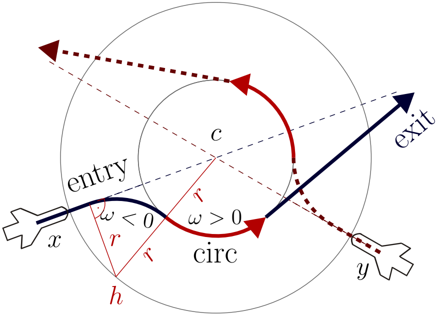


Example (d$\mathcal{L}$ formula of verification subgoal)

$$\textit{safe} \land \textit{far} \land \textit{compatible} \rightarrow [\textit{entry}](\textit{safe} \land \textit{tangential})$$

Verification Loop for Air Traffic Control

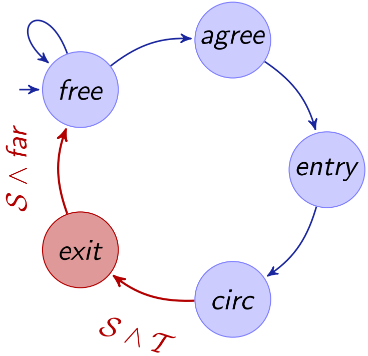


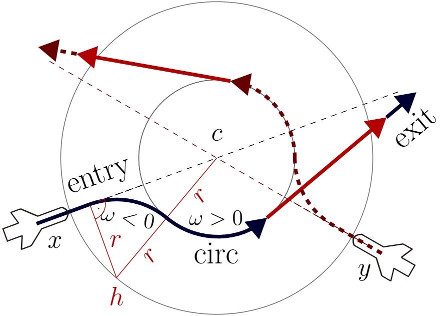


Example (d$\mathcal{L}$ formula of verification subgoal)

$$\mathit{safe} \wedge \mathit{tangential} \rightarrow [\mathit{circ}](\mathit{safe} \wedge \mathit{tangential})$$

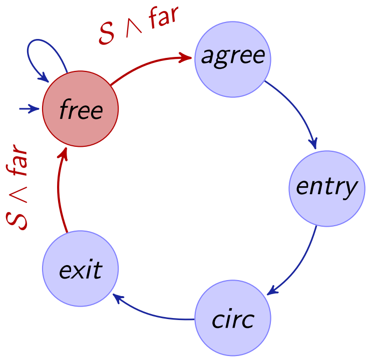
Verification Loop for Air Traffic Control

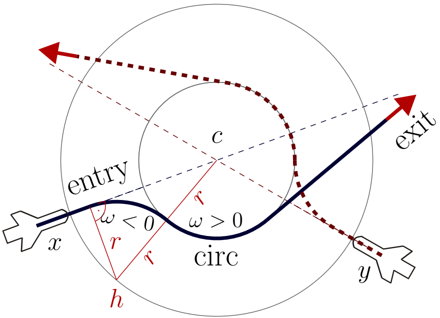




Example (d$\mathcal{L}$ formula of verification subgoal)

$$safe \land tangential \rightarrow [exit](safe \land far)$$

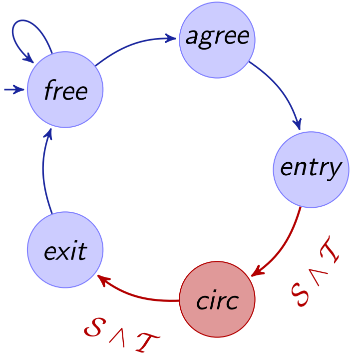


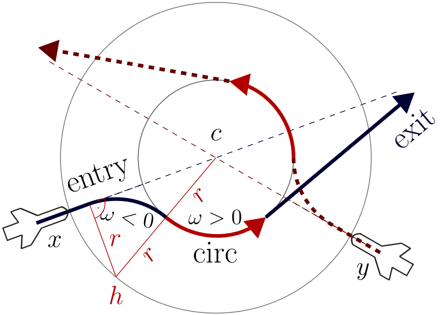


Example (d$\mathcal{L}$ formula of verification subgoal)

$$safe \wedge far \rightarrow [free](safe \wedge far)$$

Verification Loop for Air Traffic Control



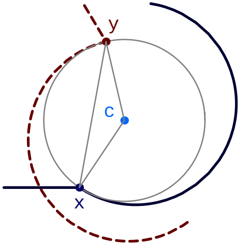


Example (d$\mathcal{L}$ formula of verification subgoal)

$$\mathit{safe} \wedge \mathit{tangential} \;\rightarrow\; [\mathit{circ}](\mathit{safe} \wedge \mathit{tangential})$$

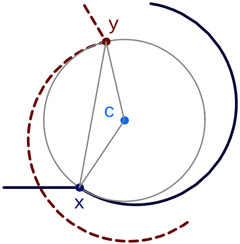
Verify Roundabout Flight with Differential Invariants

$$\overline{[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \ldots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2}$$



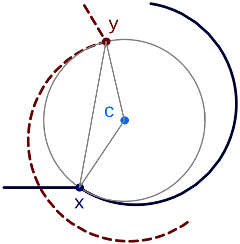
Verify Roundabout Flight with Differential Invariants

$$\frac{\frac{\partial\|x-y\|^2}{\partial x_1}x_1' + \frac{\partial\|x-y\|^2}{\partial y_1}y_1' + \frac{\partial\|x-y\|^2}{\partial x_2}x_2' + \frac{\partial\|x-y\|^2}{\partial y_2}y_2' \geq \frac{\partial p^2}{\partial x_1}x_1' \dots}{[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \dots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2}$$

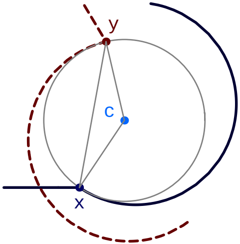


Verify Roundabout Flight with Differential Invariants

$$\frac{\frac{\partial\|x-y\|^2}{\partial x_1}x_1' + \frac{\partial\|x-y\|^2}{\partial y_1}y_1' + \frac{\partial\|x-y\|^2}{\partial x_2}x_2' + \frac{\partial\|x-y\|^2}{\partial y_2}y_2' \geq \frac{\partial p^2}{\partial x_1}x_1' \ldots}{[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \ldots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2}$$



$$\frac{\dfrac{\partial\|x-y\|^2}{\partial x_1}d_1 + \dfrac{\partial\|x-y\|^2}{\partial y_1}e_1 + \dfrac{\partial\|x-y\|^2}{\partial x_2}d_2 + \dfrac{\partial\|x-y\|^2}{\partial y_2}e_2 \geq \dfrac{\partial p^2}{\partial x_1}d_1 \ldots}{[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \ldots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2}$$

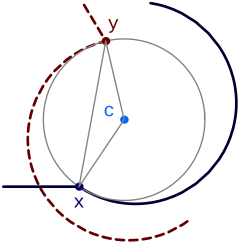


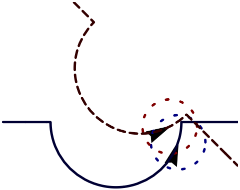
Verify Roundabout Flight with Differential Invariants

$$2(x_1 - y_1)(d_1 - e_1) + 2(x_2 - y_2)(d_2 - e_2) \geq 0$$

$$\frac{\partial \|x-y\|^2}{\partial x_1} d_1 + \frac{\partial \|x-y\|^2}{\partial y_1} e_1 + \frac{\partial \|x-y\|^2}{\partial x_2} d_2 + \frac{\partial \|x-y\|^2}{\partial y_2} e_2 \geq \frac{\partial p^2}{\partial x_1} d_1 \dots$$

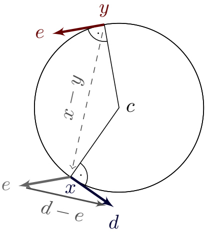
$$[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \dots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2$$

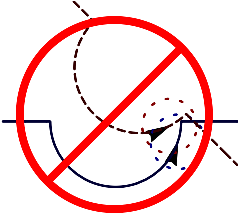




Verify Roundabout Flight with Differential Invariants

$$\dfrac{\dfrac{2(x_1 - y_1)\color{red}{(d_1 - e_1)} + 2(x_2 - y_2)\color{red}{(d_2 - e_2)} \geq 0}{\frac{\partial\|x-y\|^2}{\partial x_1} d_1 + \frac{\partial\|x-y\|^2}{\partial y_1} e_1 + \frac{\partial\|x-y\|^2}{\partial x_2} d_2 + \frac{\partial\|x-y\|^2}{\partial y_2} e_2 \geq \frac{\partial p^2}{\partial x_1} d_1 \ldots}}{[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \ldots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2}$$





$$\overline{[d_1' = -\omega d_2, e_1' = -\omega e_2, x_2' = d_2, d_2' = \omega d_1 ..]\color{red}{d_1 - e_1 = -\omega(x_2 - y_2)}}$$

Verify Roundabout Flight with Differential Invariants

$$\cfrac{\cfrac{2(x_1-y_1)(d_1-e_1)+2(x_2-y_2)(d_2-e_2)\geq 0}{\frac{\partial\|x-y\|^2}{\partial x_1}d_1+\frac{\partial\|x-y\|^2}{\partial y_1}e_1+\frac{\partial\|x-y\|^2}{\partial x_2}d_2+\frac{\partial\|x-y\|^2}{\partial y_2}e_2\geq\frac{\partial p^2}{\partial x_1}d_1\dots}}{[x_1'=d_1,d_1'=-\omega d_2,x_2'=d_2,d_2'=\omega d_1\dots](x_1-y_1)^2+(x_2-y_2)^2\geq p^2}$$

Proposition (Differential saturation)

F differential invariant of $x'=\theta\wedge H$, then
$x'=\theta\wedge H$ equivalent to $x'=\theta\wedge H\wedge F$

$$\cfrac{}{[d_1'=-\omega d_2,e_1'=-\omega e_2,x_2'=d_2,d_2'=\omega d_1..]d_1-e_1=-\omega(x_2-y_2)}$$

Verify Roundabout Flight with Differential Invariants

$$2(x_1 - y_1)(-\omega(x_2 - y_2)) + 2(x_2 - y_2)\omega(x_1 - y_1) \geq 0$$

$$2(x_1 - y_1)(d_1 - e_1) + 2(x_2 - y_2)(d_2 - e_2) \geq 0$$

$$\frac{\partial\|x-y\|^2}{\partial x_1} d_1 + \frac{\partial\|x-y\|^2}{\partial y_1} e_1 + \frac{\partial\|x-y\|^2}{\partial x_2} d_2 + \frac{\partial\|x-y\|^2}{\partial y_2} e_2 \geq \frac{\partial p^2}{\partial x_1} d_1 \ldots$$

$$[x_1' = d_1, d_1' = -\omega d_2, x_2' = d_2, d_2' = \omega d_1 \ldots](x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2$$

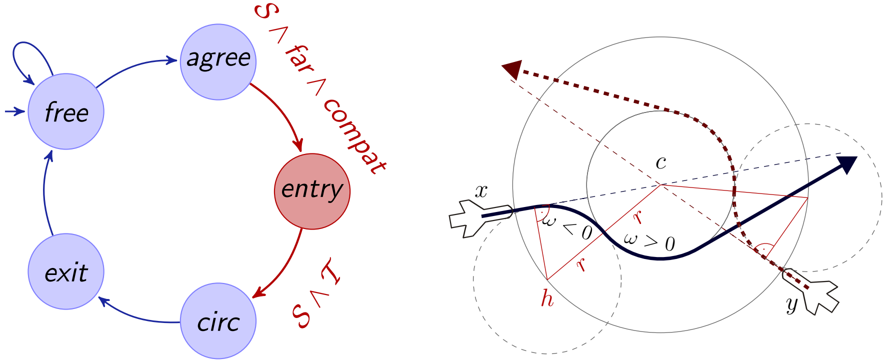
Proposition (Differential saturation)

F differential invariant of $x' = \theta \land H$, then

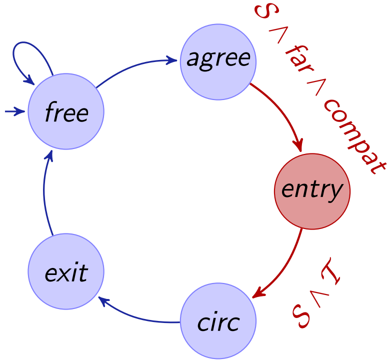
$$x' = \theta \land H \quad \text{equivalent to} \quad x' = \theta \land H \land F$$

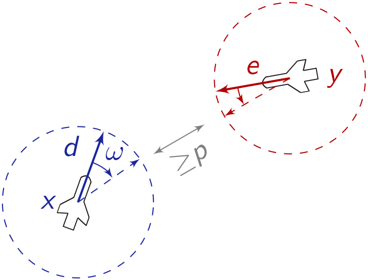
$$[d_1' = -\omega d_2, e_1' = -\omega e_2, x_2' = d_2, d_2' = \omega d_1 ..] d_1 - e_1 = -\omega(x_2 - y_2)$$

Flyable Roundabout Maneuver: Entry

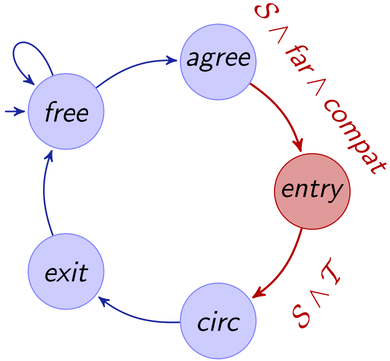


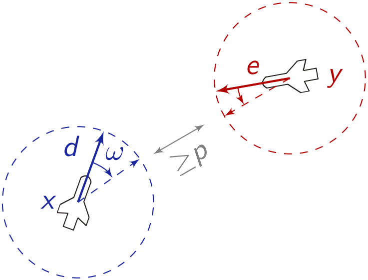
Flyable Roundabout Maneuver: Entry



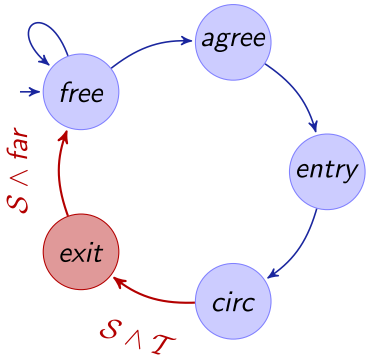


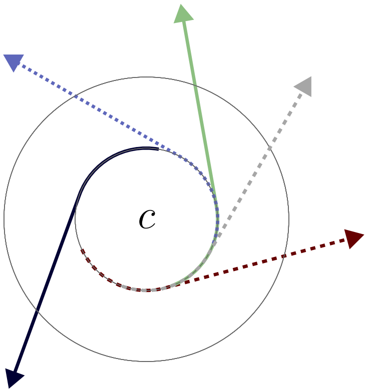
Flyable Roundabout Maneuver: Entry



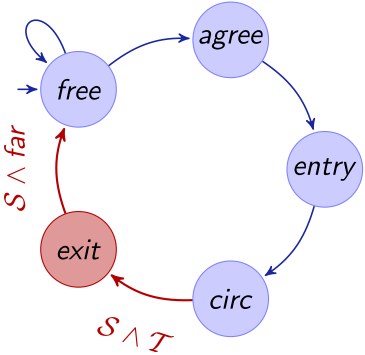


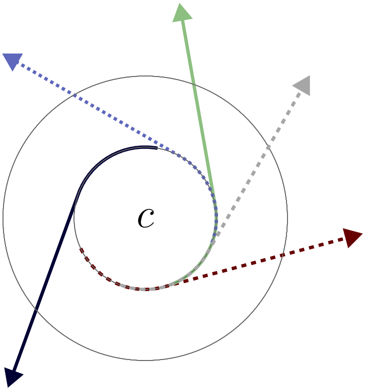
Flyable Roundabout Maneuver: Exit



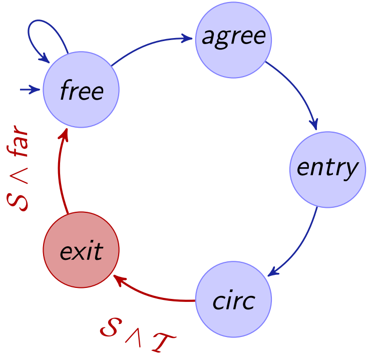


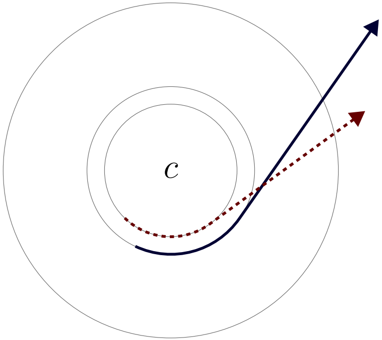
Flyable Roundabout Maneuver: Exit



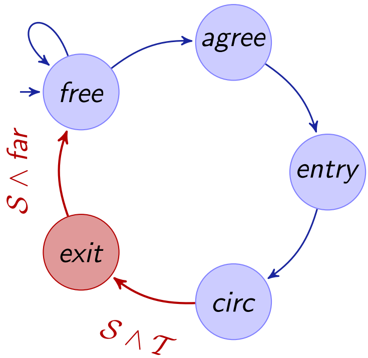


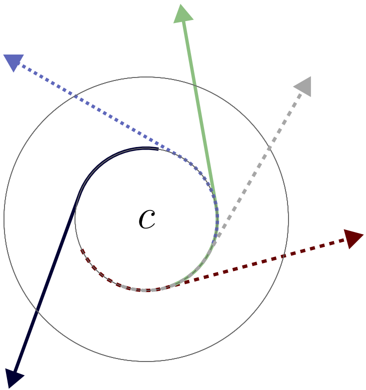
Flyable Roundabout Maneuver: Exit





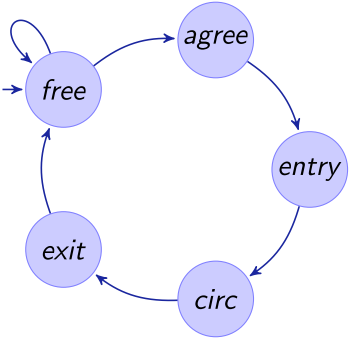
Flyable Roundabout Maneuver: Exit

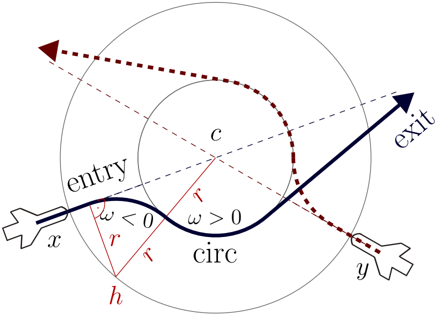




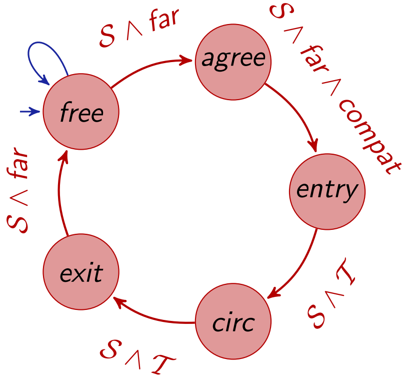
Outline

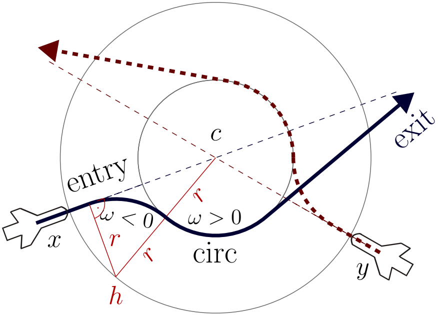
Flyable Roundabout Maneuver: Summary



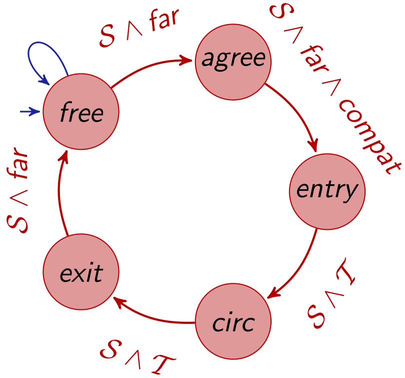


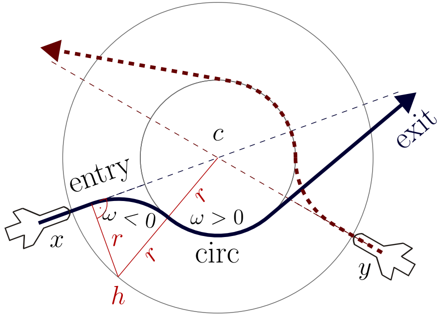
Flyable Roundabout Maneuver: Summary





Flyable Roundabout Maneuver: Summary





Theorem (Collision freedom)

FTRM is collision free:

$$\|x - y\| \geq far \land \ldots \to [FTRM]\|x - y\| \geq p$$

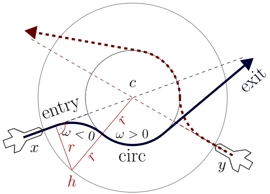
Outline

Experimental Results

Case Study	Time(s)	Mem(Mb)	Steps	Dim
tangential roundabout (2a/c)	10.4	6.8	197	13
tangential roundabout (3a/c)	253.6	7.2	342	18
tangential roundabout (4a/c)	382.9	10.2	520	23
tangential roundabout (5a/c)	1882.9	39.1	735	28
bounded maneuver speed	0.5	6.3	14	4
flyable roundabout entry*	10.1	9.6	132	8
flyable entry feasible*	104.5	87.9	16	10
flyable entry circular	3.2	7.6	81	5
limited entry progress	1.9	6.5	60	8
entry separation	140.1	20.1	512	16
mutual negotiation successful	0.8	6.4	60	12
mutual negotiation feasible*	7.5	23.8	21	11
mutual far negotiation	2.4	8.1	67	14
simultaneous exit separation*	4.3	12.9	44	9
different exit directions	3.1	11.1	42	11

Outline

Conclusions & Future Work



- Formal verification can scale to real aircraft maneuvers!
- Differential invariants instead of reachability along solutions
- Fixedpoint computations to find differential invariants
- Compositional verification
- Challenging arithmetic complexity (simplifications)

- Improve differential invariant generation
- Abstract interpretation domain
- Widening in fixedpoint loop
- Nonlinear real arithmetic